A Modification of Roulier's Algorithm for Shape-preserving Surface Interpolation

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A Modification of Roulier’s Algorithm for Shape-preserving Surface Interpolation.

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ABSTRACT

This paper considers the problem of shape-preserving interpolation for grid data. An earlier algorithm due to Roulier [18], based on a shape-preserving curve interpolation scheme due to McAllister and Roulier [17], is modified to use a simpler but equivalent curve interpolation scheme due to Schumaker [19].

1. Introduction

A common problem occurring in generating smooth curves or surfaces which interpolate given data is that the resulting interpolant fails to satisfy some of the user’s qualitative requirements. For example, the given data may be from a surface which is known to be monotone and/or convex, but the resulting interpolant may not satisfy these properties and may induce artificial or exaggerated hills and valleys in the interpolating surface. A few methods have been developed which attempt to satisfy such requirements, amongst which are the so-called shape-preserving methods. These methods are shape-preserving in the sense that if the data exhibits a given monotonicity and/or convexity along all grid lines parallel to the axes, then the resulting approximation also exhibits the same monotonicity and/or convexity along all lines parallel to these grid lines as well.

Shape-preserving interpolation has a wide variety of applications in areas such as computer-aided geometric design, data analysis, and mathematical modelling. For the case of curve generation, several methods have been proposed by a number of authors which preserve properties such as non-negativity, monotonicity and/or convexity of the
data (see, for example, references in [2, 7, 8, 12, 16, 19]). However, shape-preserving interpolation techniques for the surface generation problem are not yet adequately dealt with, and only a few methods are available, most of which are global. Dodd, McAllister and Roulier [11] describe a scheme in which the one-dimensional shape-preserving method of McAllister and Roulier [17] is applied to estimate the functions and the normal derivatives along the grid lines, and then rectangular patches are constructed using blending functions as proposed by Gregory [15]. Their algorithm produces visually pleasing results, but may not preserve the shape of the data inside the rectangles in extreme cases. Beatson and Ziegler [3] give a method which preserves the monotonicity of the data by a $C^1$ piecewise quadratic function defined over triangular elements which are obtained by subdividing each mesh rectangle into a grid of sixteen triangles. The quadratic polynomials are uniquely determined by the function value and first partial derivatives at the vertices of the grid rectangle. A similar algorithm is presented by Asaturyan and Unsworth [1], where monotonicity is achieved using biquadratic splines defined over a rectangular elements formed by the partition of each initial rectangle into four subrectangles. The biquadratic functions are determined by the values of the function, its first partial derivatives, and its cross derivative (twist) at the mesh points. Carlson and Fritsch [5, 6] have extended their univariate work on monotone interpolation [13, 14] to surfaces in the case where the data points lie on a rectangular mesh. The interpolating function is a bicubic polynomial on each panel of the data mesh, with first derivative continuity across the panel boundaries. The function is described by its function value, first partial derivative parallel to each axis and the first cross derivative, at each point of the mesh. Costantini and Fontanella [10] have approached the bivariate shape-preserving interpolation problem in a different way, by local adjustment of the degree of the interpolating functions (in either of the variables). No constraints are imposed on the partial derivatives as in above methods. They make use of tensor product formulation to adapt their univariate shape-preserving interpolation methods [7, 8, 9] to the bivariate case and the resulting interpolant is of an arbitrary continuity class.

In contrast to above methods, Roulier [18] presents a refinement technique which is
local and successively refines grid data which is convex along grid lines in such a way that
the refined data exhibit the same convexity and monotonicity along the appropriate grid
lines. This report discusses the use of the Butland [4] slope estimation technique in the
method of Roulier [18]. In section 3 we review the Roulier algorithm briefly and present a
modified algorithm in section 4 that illustrates the ways in which Butland slopes and the
one-dimensional shape-preserving method of Schumaker [19] can be used in the Roulier
method. Some numerical results are shown in section 5 and finally conclusions are given
in section 6.

2. Definitions and Notation

Let a rectangular grid data $\{(x_i, y_j, f_{i,j}) : \ i = 1, ..., n; \ j = 1, ..., m\}$ be given such
that $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_m$. Now we introduce some notation and
definitions as follows.

$$\delta x_{i,j} = (f_{i+1,j} - f_{i,j})/(x_{i+1} - x_i), \ i = 1, ..., n-1; \ j = 1, ..., m$$
$$\delta y_{i,j} = (f_{i,j+1} - f_{i,j})/(y_{j+1} - y_j), \ i = 1, ..., n-1; \ j = 1, ..., m.$$

Monotonicity: The data is said to be monotone decreasing (monotone increasing)
along the grid line $x = x_i$ if $\delta y_{i,j} \leq 0$ ($\delta y_{i,j} \geq 0$), $j = 1, ..., m-1$.

Convexity: The data is said to be concave (convex) along the grid line $x = x_i$ if
$\delta y_{i,j+1} \leq \delta y_{i,j}$ ($\delta y_{i,j+1} \geq \delta y_{i,j}$), $j = 1, ..., m-2$. Similarly, we can define the terms
monotone decreasing/increasing, and concave/convex along the grid line $y = y_j$.

The problem we intend to solve in this report is that of determining a surface which
interpolates the given data and which preserves convexity and monotonicity along grid
lines. Mathematically, the problem is that of finding a smooth bivariate function $P$ with
continuous first partial derivatives such that

$$P(x_i, y_j) = f_{i,j}, \ i = 1, ..., n; \ j = 1, ..., m. \quad (2.1)$$

subject to the shape-preserving constraints. In order that the interpolating function $P$
be monotonicity preserving along the grid line $y = y_j$, we must have
\[ sgn(P_x(x, y)|_{y=y_j} = sgn(\delta x_{i,j}), \quad x \in [x_i, x_{i+1}] \]

where \( P_x(x, y) \) denotes the partial derivative of \( P(x, y) \) with respect to \( x \). We say that \( P \) is convexity preserving along the grid line \( y = y_j \) if \( P_x(x, y_j) > 0 \) for \( x_i \leq x_{i+1} \). Similarly we can define the conditions for monotonicity and convexity preserving along the grid line \( x = x_i \) in terms of \( P_y(x, y) \). The interpolant \( P \) is called shape-preserving if it is both monotonicity and convexity preserving on each grid line.

3. Roulier Algorithm

Roulier [18] describes the following algorithm for convex bivariate grid data. For brevity and simplicity, we assume throughout the remainder of this report that grid data is convex and monotone increasing in both \( x \) and \( y \) directions i.e.,

\[
0 < \delta x_{i,j} < \delta x_{i+1,j}, \quad i = 1, ..., n - 2, \quad j = 1, ..., m. \tag{3.1}
\]

\[
0 < \delta y_{i,j} < \delta y_{i,j+1}, \quad i = 1, ..., n, \quad j = 1, ..., m - 2. \tag{3.2}
\]

and the extension to convex decreasing grid data is trivial.

In the first step of the method, one dimensional shape-preserving splines are found for the sets of data \( \{(x_i, y_j, f_{i,j}) : i = 1, ..., n\} \) for each \( j \). These one-dimensional approximations are used to determine an approximate value for the function at the midpoints of each interval on grid lines in the \( x \) directions. At the next step the roles of \( x \) and \( y \) are reversed. The typical one-dimensional shape-preserving interpolation step on a grid line is characterised by data of the form \((t_k, f_k), k = 1, ..., n\), where \( t_k \) is either \( x_k \) or \( y_k \), and where

\[
0 < \delta_1 < \delta_2 < \ldots < \delta_{n-1} \tag{3.3}
\]

and

\[
\delta_k = (f_{k+1} - f_k)/(t_{k+1} - t_k), \quad k = 1, ..., n - 1. \tag{3.4}
\]

The step computes an estimates \( d_k \) of the slopes at \( t_k \) which satisfy

\[
\delta_k < d_{k+1} < \delta_{k+1}, \quad k = 1, ..., n - 2, \quad d_1 < \delta_1, \quad \delta_{n-1} < d_n \tag{3.5}
\]

4
In terms of these slopes, we define constant $A_k, B_k$ as follows

$$A_1 = d_2(t_1 - t_2) + f_2$$

$$A_k = \max(d_{k-1}(t_k - t_{k-1}) + f_{k-1}, \ d_k(t_k - t_{k+1}) + f_{k+1}), \quad k = 2, \ldots, n-1.$$ 

$$A_n = d_{n-1}(t_n - t_{n-1}) + f_{n-1}$$

and

$$B_k = \max(d_k(\bar{t}_k - t_k) + f_k, \ d_k(\bar{t}_k - t_{k+1}) + f_{k+1}), \quad k = 1, \ldots, n-1.$$ 

where $\bar{t}_k = \frac{(t_k + t_{k+1})}{2}$.

The algorithm is presented by the following Pascal-like pseudocode.

**Step 1**

For $j := 1$ to $m$ do

Begin

For $i := 1$ to $n$ do

Use the slopes calculation method as proposed by the McAllister and Roulier to produce estimates of slopes $dx_{i,j}$.

For $i := 1$ to $n$ do

Use the shape-preserving quadratic interpolant of the McAllister and Roulier to produce $P_j$ such that $P_j(x_i) = f_{i,j}$ and $P'_j(x_i) = dx_{i,j}$.

End

**Step 2**

For $j := 1$ to $m$ do

For $i := 1$ to $n - 1$ do

Begin

$$\bar{x}_i = \frac{(x_i + x_{i+1})}{2}; \quad f_{i,j}^* = P_j(x_i); \quad \bar{f}_{i,j} = \frac{(f_{i,j} + f_{i+1,j})}{2}$$

End

For $j := 1$ to $m$ do

For $i := 1$ to $n - 1$ do

Begin

Generate numbers $B_{i,j}$ (corresponding to $B_k$ above) with $dx_{i,j}$ from step 1 and check that they satisfy $B_{i,j} < f_{i,j}^* < \bar{f}_{i,j}$.

End
Step 3
For $i := 1$ to $n - 1$ do

For $j := 2$ to $m$ do

Begin

$\Delta y_{i,j} = \frac{(f_{i,j} - f_{i,j-1})}{(y_j - y_{j-1})}$

End

Step 4

If for some $i_0$ we have a $j_0$ such that $\Delta y_{i_0,j_0} > \Delta y_{i_0,j_0+1}$, then use the data $(y_j, f_{i_0,j})$ to generate $A_{i_0,j}$, $j = 1, \ldots, m$ (corresponding to $A_k$ above).

Step 5

For this $i_0$ if $f_{i_0,j} < A_{i_0,j}$ then set $f_{i_0,j} = A_{i_0,j}$. This guarantees that $(y_j, f_{i_0,j})$, $j = 1, \ldots, m$ are convex.

Step 6

Go to Step 1 and use new grid data consisting of $(x_i, y_j, f_{i,j})$ $i = 1, \ldots, n$; $j = 1, \ldots, m$ and $(\bar{x}_i, y_j, f_{i,j}^*)$ $i = 1, \ldots, n - 1$; $j = 1, \ldots, m$ but reverse the roles of i and j.

The points on a surface are generated and monotonicity and convexity parallel to the grid lines is maintained by the algorithm when applied to convex grid data.

4. Modified Algorithm

Now we present a new modified algorithm based on an alternative one-dimensional shape-preserving algorithm due to Schumaker, using a slope estimation technique due to Butland. At the interior data points $(t_k, f_k)$, $k = 2, \ldots, n - 1$, the slopes $d_k$ are calculated as

$$d_k = \begin{cases} \frac{2\delta_{k-1}\delta_k}{(\delta_{k-1}+\delta_k)}, & \text{if } \delta_{k-1}\delta_k > 0 \\ 0, & \text{otherwise} \end{cases} \quad (4.1)$$
These satisfy (3.5), since from (3.3), we have

\[
\delta_k < \delta_{k+1} \\
\delta_k^2 < \delta_k \delta_{k+1} \\
\delta_k^2 + \delta_k \delta_{k+1} < 2\delta_k \delta_{k+1} \\
\delta_k < \frac{2\delta_k \delta_{k+1}}{(\delta_k + \delta_{k+1})} \\
\delta_k < d_{k+1}
\]

(4.2)

and also from (4.2),

\[
\delta_k \delta_{k+1} < \delta_{k+1}^2 \\
2\delta_k \delta_{k+1} < \delta_k \delta_{k+1} + \delta_{k+1}^2 \\
\frac{2\delta_k \delta_{k+1}}{(\delta_k + \delta_{k+1})} < \delta_{k+1} \\
d_{k+1} < \delta_{k+1}
\]

(4.4)

Now by (4.3) and (4.4), we obtain (3.5)

Schumaker [19] describes a quadratic spline interpolation method which does not seem to be shape-preserving (in one pass) and often the user is required to adjust the slopes interactively to ensure the monotonicity and convexity conditions. However Iqbal [16] has shown that Schumaker’s method becomes a one pass method automatically if the Butland slopes are used and that in this case it produces an interpolant which is identical with McAllister and Roulier interpolant, though by an easier computation. The bivariate interpolant produced by the modified algorithm given below is therefore identical.

In the modified algorithm step 1 of the main algorithm is replaced using the Butland [4] slopes and the Schumaker [19] shape-preserving interpolant in such a manner that resulting interpolant preserves the convexity and monotonicity of the grid data. Now step 1 can be described as
For $j := 1$ to $m$ do

Begin

For $i := 1$ to $n$ do

Use the Butland slope formulae to produce slopes $dx_{i,j}$.

For $i := 1$ to $n$ do

Use shape-preserving quadratic interpolation method of the Schumaker to produce $P_j$ such that $P_j(x_i) = f_{i,j}$ and $P'_j(x_i) = dx_{i,j}$.

End.

5. Numerical Results

The algorithm described in the previous section has been tested on several sets of grid data taken from the literature, but here we consider only two convex data sets used earlier by Roulier [18] to illustrate the performance of the method, and the corresponding plots of the interpolants are shown in figures 5.1 and 5.2 respectively. These examples reveal that both algorithms give identical interpolants and the resulting surfaces are shape-preserving. We conclude from these figures that the modified algorithm produces "visually pleasing" surfaces that are identical to the the surfaces given by Roulier’s method.

6. Conclusions

In conclusion, we want to point out that our modified algorithm provides an alternative and simpler method for constructing the interpolant given by Roulier’s method. The modified algorithm has the merit that it is very easy to implement, and that it is more efficient in terms of both CPU time and storage requirements.

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